

THE SHARP UPPER BOUNDS FOR THE FIRST POSITIVE EIGENVALUE OF KOHN-LAPLACIAN ON COMPACT STRICTLY PSEUDOCONVEX HYPERSURFACES

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ABSTRACT. We give sharp and explicit upper bounds for the first positive eigenvalue of $\lambda_1(\square_b)$ of the Kohn-Laplacian on compact strictly pseudoconvex hypersurfaces in \mathbb{C}^{n+1} in terms of their defining functions. As an application, we show that in the family of real ellipsoids, $\lambda_1(\square_b)$ has a unique maximum value at the CR sphere.

1. INTRODUCTION

Let (M^{2n+1}, θ) be a compact strictly pseudoconvex pseudohermitian manifold of real dimension $2n + 1 \geq 3$. Let $\bar{\partial}_b: L^2(M) \rightarrow L^2_{0,1}(M)$ be the tangential Cauchy–Riemann operator and $\bar{\partial}_b^*$ the formal adjoint with respect to the volume measure $dv = \theta \wedge (d\theta)^n$. The Kohn-Laplacian acting on functions is given by $\square_b = \bar{\partial}_b^* \bar{\partial}_b$ and the sub-Laplacian is given by $\Delta_b = 2\text{Re } \square_b$. There has been growing interest in the relation between the spectra of the sub-Laplacian and Kohn-Laplacian and the geometric qualities of the underlying CR manifolds. We mention here, for example, the Lichnerowicz-type estimate for the first positive eigenvalue of the sub-Laplacian on compact manifolds with a lower bound on Ricci and torsion was studied in, e.g., [1, 2, 13, 18, 11]. The characterization of extremal case, the Obata-type problem, was studied in, e.g., [10, 21, 16]. In particular, Wang and the first author proved an Obata-type theorem in CR geometry for compact manifolds in [21] which characterizes the CR sphere (among compact manifolds) as the only extremal case in the Lichnerowicz-type estimate for sub-Laplacian. We refer the reader to the aforementioned papers and reference therein for more details on these problems.

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The eigenvalue problem for \square_b is more involved. It is well-known that on a *non-embeddable* compact strictly pseudoconvex manifold of three-dimension, $\text{Spec}(\square_b)$ contains a sequence of “small” eigenvalues converging rapidly to zero. In this case, we can not define the first positive eigenvalue of \square_b . In fact, by the theorems of Boutet de Monvel, Burns, and Kohn, zero is an isolated eigenvalue of \square_b if and only if M is embeddable into some complex space \mathbb{C}^N [5, 4, 15]; see also [6]. Thus, for embedded manifolds, it makes sense to define and study the first positive eigenvalue λ_1 of \square_b .

In [7], Chanillo, Chiu, and Yang proved a Lichnerowicz-type lower bound for λ_1 for three-dimensional manifolds (which are not assumed to be embedded *a priori*). Their method also gives the estimate for five dimensional case. In a preprint [8], Chang and Wu gave a lower bound in general dimension and proved some partial results on characterizing the equality case. In [20], the Wang, the first, and the third author completely analyzed the equality case by establishing an Obata-type theorem for Kohn-Laplacian; we refer to [20] for more details.

In this paper, we shall give sharp *upper* bounds for λ_1 on compact strictly pseudoconvex CR manifolds embedded in \mathbb{C}^{n+1} . Suppose ρ is a smooth, strictly plurisubharmonic function on \mathbb{C}^{n+1} and ν is a regular value of ρ such that $M := \rho^{-1}(\nu)$ is compact. On M , we consider the “usual” pseudohermitian structure θ “induced” by ρ :

$$\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho), \quad (1.1)$$

where $\iota^*: M \rightarrow \mathbb{C}^{n+1}$ is the usual embedding. This pseudohermitian structure gives rise to a volume form $dv = \theta \wedge (d\theta)^n$ on M . Furthermore, ρ induces a Kähler metric $\rho_{j\bar{k}} dz^j d\bar{z}^k$ in a neighborhood U of M . Let $[\rho^{j\bar{k}}]^t$ be the inverse of $H(\rho)$. For a smooth function u on U , the length of ∂u in the Kähler metric is given by

$$|\partial u|_\rho^2 = \rho^{j\bar{k}} u_j \bar{u}_{\bar{k}}. \quad (1.2)$$

Here we use the usual summation convention: repeated Latin indices are summing from 1 to $n+1$. We also use $\rho^{j\bar{k}}$ and $\rho_{j\bar{k}}$ to raise and lower the indices, e.g., $u^{\bar{k}} = \rho^{l\bar{k}} u_l$, so that $|\partial u|_\rho^2 = \bar{u}_{\bar{k}} u^k$. We define the following degenerate differential operator

$$\tilde{\Delta}_\rho = \left(|\partial\rho|_\rho^{-2} \rho^j \rho^{\bar{k}} - \rho^{j\bar{k}} \right) \partial_j \partial_{\bar{k}}. \quad (1.3)$$

Our first result in this paper is the following sharp upper bound for λ_1 .

Theorem 1.1. *Let ρ be a smooth strictly plurisubharmonic function defined on an open set U of \mathbb{C}^{n+1} , M a compact connected regular level set of ρ , and λ_1 the first positive eigenvalue of \square_b on M . Assume that for some j ,*

$$\text{Re } \rho_{\bar{j}} \tilde{\Delta}_\rho \rho_j + \frac{1}{n} |\partial\rho|_\rho^2 |\tilde{\Delta}_\rho \rho_j|^2 \leq 0 \quad \text{on } M. \quad (1.4)$$

Then

$$\lambda_1(M, \theta) \leq n \max_M |\partial\rho|_\rho^{-2} \quad (1.5)$$

and the equality holds only if $|\partial\rho|_\rho^2$ is constant along M .

The upper bound in (1.5) is sharp and the equality occurs on the sphere with the standard pseudohermitian structure. Moreover, in Example 4.3 below, we shall see that the condition (1.4) can not be relaxed.

Notice that condition (1.4) is satisfied if there exists j such that $\rho_{j\bar{k}l} = 0$ for all k and l and hence we can easily construct examples for which Theorem 1.1 does apply. In particular, if $\rho_{j\bar{k}} = \delta_{jk}$, then (1.4) holds. We shall show that in this case, we can improve the estimate by taking the average value of $|\partial\rho|_\rho^{-2}$ instead of its maximum. Thus, we define $v(M) = \int_M \theta \wedge (d\theta)^n$ be the volume of M .

Theorem 1.2. *Let ρ be a smooth strictly plurisubharmonic function defined on an open set U of \mathbb{C}^{n+1} , M a compact connected regular level set of ρ , and λ_1 the first positive eigenvalue of \square_b on M . Suppose that $\rho_{j\bar{k}} = \delta_{jk}$, then*

$$\lambda_1 \leq \frac{n}{v(M)} \int_M |\partial\rho|_\rho^{-2} \theta \wedge (d\theta)^n. \quad (1.6)$$

The equality occurs only if $|\partial\rho|_\rho^2$ is a constant on M . If furthermore, ρ is defined in the domain bounded by M , then M must be a sphere.

The estimate (1.6) is a special case of a more general estimate in Theorem 4.1 below which provides a sharp upper bound for λ_1 in terms of the eigenvalues of the complex Hessian matrix $[\rho_{j\bar{k}}]$. In another direction, we shall prove estimate (1.6) for a larger class of CR manifold in Theorem 6.1 in Section 6.

Our main motivation to prove the upper bound in Theorem 1.2 comes from its application to eigenvalue problems on the real *ellipsoids*, the compact regular level sets of a real plurisubharmonic quadratic polynomial. The ellipsoids was studied by Webster [24] who showed that an ellipsoid is not biholomorphic equivalent to the sphere unless it is complex linearly equivalent to the sphere. (It is now well-known that two generic ellipsoids are not biholomorphic equivalent). The eigenvalue problem on ellipsoids was also studied by Tran and the first author [22]. This paper provides an upper bound for the first positive eigenvalue of Δ_b on the real ellipsoids in \mathbb{C}^2 . We shall show that on real ellipsoids, the upper bound in Theorem 1.2 can be computed explicitly.

Corollary 1.3. *Let $\rho(Z)$ be a real-valued, strictly plurisubharmonic homogeneous quadratic polynomial satisfying $\rho_{j\bar{k}} = \delta_{jk}$. Suppose that $M = \rho^{-1}(\nu)$ ($\nu > 0$) is a compact*

connected regular level sets of ρ . Then

$$\lambda_1(M, \theta) \leq \lambda_1(\sqrt{\nu} \mathbb{S}^{2n+1}, \theta_0) = n/\nu. \quad (1.7)$$

The equality occurs if and only if $(M, \theta) = (\sqrt{\nu} \mathbb{S}^{2n+1}, \theta_0)$.

Here, $\sqrt{\nu} \mathbb{S}^{2n+1}$ is the sphere $\|Z\|^2 = \nu$ and $\theta_0 = \iota^*(i/2)\bar{\partial}\|Z\|^2$ is the “standard” pseudohermitian structure on the sphere.

The paper is organized as follows. In Section 2, we shall give two simple formulas for Kohn-Laplacian on compact real hypersurfaces in complex manifolds. These formulas allow us to compute \square_b explicitly in terms of the defining function ρ ; see Proposition 2.1. These formulas will be crucial for latter sections. In Section 3, we shall prove a general estimate for $\lambda_1(\square_b)$ and Theorem 1.1. In Section 4, we shall give a sharp upper bound for λ_1 in terms of the eigenvalues of the complex Hessian $[\rho_{j\bar{k}}]$, implying the estimate in Theorem 1.2, and prove the characterization of equality case. We also give a family of examples (beside the ellipsoids) where we can apply this bound. These examples also show that the condition (1.4) in Theorem 1.1 can not be relaxed. In Section 5, we shall compute the bound in Theorem 1.2 explicitly in the case of ellipsoids, proving Corollary 1.3. In Section 6, we prove the estimate (1.6) for the manifolds imbedded into the CR sphere.

2. KOHN LAPLACIAN ON COMPACT REAL HYPERSURFACES

In this section, we shall give two formulas for \square_b on a compact regular level set of a Kähler potential ρ in terms of $\partial\rho$ and the metric $\rho_{j\bar{k}}dz^jdz^{\bar{k}}$. First, let us start with a compact real hypersurface in \mathbb{C}^{n+1} arising as a regular level set of a strictly plurisubharmonic function ρ :

$$M = \rho^{-1}(\nu) := \{Z \in U : \rho(Z) = \nu\}. \quad (2.1)$$

Here ρ is smooth on a neighborhood U of M and $d\rho \neq 0$ along M . We assume that the complex Hessian $H(\rho) = [\rho_{j\bar{k}}]$ is positive definite and thus ρ defines a Kähler metric $\rho_{j\bar{k}}dz^jdz^{\bar{k}}$ on U . Let $[\rho^{j\bar{k}}]^t$ be the inverse of $H(\rho)$. For a smooth function u on U , the length of ∂u in the Kähler metric is then given by

$$|\partial u|_\rho^2 = \rho^{j\bar{k}}u_j\bar{u}_{\bar{k}}. \quad (2.2)$$

We shall always equip M with the pseudohermitian structure θ “induced” by ρ :

$$\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho). \quad (2.3)$$

For local computations, it is convenient to work in the local admissible holomorphic coframe $\{\theta^\alpha : \alpha = 1, 2, \dots, n\}$ on M given by

$$\theta^\alpha = dz^\alpha - ih^\alpha\theta, \quad h^\alpha = |\partial\rho|_\rho^{-2}\rho^\alpha = |\partial\rho|_\rho^{-2}\rho_{\bar{j}}\rho^{\alpha\bar{j}}, \quad \alpha = 1, 2, \dots, n. \quad (2.4)$$

This admissible coframe is valid when $\rho_{n+1} \neq 0$. It is shown by Luk and the first author [19, p. 679] that at the point p with $\rho_{n+1} \neq 0$,

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}, \quad (2.5)$$

where the Levi matrix $[h_{\alpha\bar{\beta}}]$ is given explicitly:

$$h_{\alpha\bar{\beta}} = \rho_{\alpha\bar{\beta}} - \rho_\alpha \partial_{\bar{\beta}} \log \rho_{n+1} - \rho_{\bar{\beta}} \partial_\alpha \log \rho_{n+1} + \rho_{n+1} \frac{\rho_\alpha \rho_{\bar{\beta}}}{|\rho_{n+1}|^2}. \quad (2.6)$$

We can check directly that the inverse $[h^{\gamma\bar{\beta}}]$ of the Levi matrix is given by

$$h^{\gamma\bar{\beta}} = \rho^{\gamma\bar{\beta}} - \frac{\rho^\gamma \rho^{\bar{\beta}}}{|\partial\rho|_\rho^2}, \quad \rho^\gamma = \sum_{k=1}^{n+1} \rho_{\bar{k}} \rho^{\gamma\bar{k}}. \quad (2.7)$$

We use the Levi matrix and its inverse to lower and raise the Greek indices; repeated Greek indices are summing from 1 to n . The Tanaka-Webster covariant derivatives are given by

$$\nabla_\alpha \nabla_{\bar{\beta}} f = Z_\alpha Z_{\bar{\beta}} f - \omega_{\bar{\beta}}^{\bar{\sigma}}(Z_\alpha) Z_{\bar{\sigma}} f \quad (2.8)$$

where $\{Z_\alpha\}$ is the holomorphic frame dual to $\{\theta^\alpha\}$ and $\omega_{\bar{\beta}}^{\bar{\sigma}}$ are the connection forms. More precisely,

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} - \frac{\rho_\alpha}{\rho_{n+1}} \frac{\partial}{\partial z_{n+1}}, \quad (2.9)$$

and the Tanaka-Webster connection forms are computed in [19]; see also [24].

$$\omega_{\bar{\beta}\alpha} = (Z_{\bar{\gamma}} h_{\alpha\bar{\beta}} - h_{\bar{\beta}} h_{\alpha\bar{\gamma}}) \theta^{\bar{\gamma}} + h_{\alpha\bar{\gamma}} h_{\gamma\bar{\beta}} \theta^\gamma + i h_{\alpha\bar{\sigma}} Z_{\bar{\beta}} h^{\bar{\sigma}} \theta, \quad h_\alpha = h_{\alpha\bar{\beta}} h^{\bar{\beta}}. \quad (2.10)$$

Also, the Reeb vector field is given by

$$T = i \sum_{j=1}^{n+1} \left(h^j \frac{\partial}{\partial z^j} - h^{\bar{j}} \frac{\partial}{\partial \bar{z}^j} \right), \quad h^j = \frac{\rho^j}{|\partial\rho|_\rho^2}. \quad (2.11)$$

The formula (2.12) below, expressing \square_b in terms of ρ , will be crucial for our analysis.

Proposition 2.1. *Let U be an open set in a Kähler manifold X and ρ a Kähler potential on U . Let M be a smooth compact, connected, regular level set of ρ , $\theta = \frac{i}{2}(\bar{\partial}\rho - \partial\rho)$, and \square_b the Kohn-Laplacian defined on M with respect to $dv = \theta \wedge (d\theta)^n$.*

(i) *If f is a smooth function on U , then the following identity holds on M .*

$$\square_b f = -\text{trace}(i\partial\bar{\partial}f) + |\partial\rho|_\rho^{-2} \langle \partial\bar{\partial}f, \partial\rho \wedge \bar{\partial}\rho \rangle + n|\partial\rho|_\rho^{-2} \langle \partial\rho, \bar{\partial}f \rangle, \quad (2.12)$$

(ii) *Suppose that $(z^1, z^2, \dots, z^{n+1})$ is a local coordinate system on an open set V . Define the vector fields*

$$X_{jk} = \rho_k \partial_j - \rho_j \partial_k, \quad X_{\bar{j}\bar{k}} = \overline{X_{jk}}. \quad (2.13)$$

Then the following holds on $M \cap V$.

$$\square_b f = -\frac{1}{2} |\partial \rho|_\rho^{-2} \rho^{p\bar{k}} \rho^{q\bar{j}} X_{pq} X_{\bar{j}\bar{k}} f. \quad (2.14)$$

Remark 2.2. (a) The trace operator is taken with respect to the Kähler form and thus $-\text{trace}(i\partial\bar{\partial}f)$ is the Laplace-Beltrami operator acting on f . In local coordinates, (2.12) can be written as

$$\square_b f = \left(|\partial \rho|_\rho^{-2} \rho^k \rho^{\bar{j}} - \rho^{\bar{j}k} \right) f_{\bar{j}k} + n |\partial \rho|_\rho^{-2} \rho^{\bar{k}} f_{\bar{k}}. \quad (2.15)$$

(b) Formulas (2.14) and (2.12) are generalizations of two formulas for Kohn-Laplacian on the sphere appeared in [12]. This paper also studies Kohn-Laplacian for forms on the sphere (with volume element induced from \mathbb{C}^{n+1}). Notice that the fields X_{jk} are *tangential* Cauchy-Riemann vector fields on M generating $T^{1,0}$ at each point.

Proof. We first prove (i). It is well-known [17] that the Kohn Laplacian acting on function can be given locally by

$$-\square_b f = h^{\bar{\beta}\alpha} \nabla_\alpha \nabla_{\bar{\beta}} f. \quad (2.16)$$

Thus, we can work in a local coordinate $(z^1, z^2, \dots, z^n, w = z^{n+1})$ on X and assume that $\rho_w = \partial_w \rho \neq 0$. Choose the local frame and coframe as above. Notice that

$$Z^{\bar{\beta}} = h^{\alpha\bar{\beta}} Z_\alpha = h^{\alpha\bar{\beta}} \partial_\alpha - h^{\alpha\bar{\beta}} \frac{\rho_\alpha}{\rho_{n+1}} \partial_{n+1} = \rho^{k\bar{\beta}} \partial_k - \frac{\rho^{\bar{\beta}}}{|\partial \rho|_\rho^2} \rho^k \partial_k. \quad (2.17)$$

Therefore,

$$\begin{aligned} -\square_b f &= Z^{\bar{\beta}} Z_\beta f - n h^{\bar{\sigma}} f_{\bar{\sigma}} \\ &= \left[\rho^{k\bar{\beta}} \partial_k - |\partial \rho|_\rho^{-2} \rho^{\bar{\beta}} \rho^k \partial_k \right] \left[f_{\bar{\beta}} - \frac{\rho_{\bar{\beta}}}{\rho_{\bar{w}}} f_{\bar{w}} \right] - n h^{\bar{\sigma}} f_{\bar{\sigma}} \\ &= \rho^{k\bar{\beta}} f_{\bar{\beta}k} - \frac{\rho_{\bar{\beta}} \rho^{k\bar{\beta}} f_{\bar{w}k}}{\rho_{\bar{w}}} - \rho^{k\bar{\beta}} f_{\bar{w}} \left[\frac{\rho_{\bar{w}} \rho_{\bar{\beta}k} - \rho_{\bar{\beta}} \rho_{\bar{w}k}}{\rho_{\bar{w}}^2} \right] \\ &\quad - \frac{\rho^k \rho^{\bar{\beta}} f_{\bar{\beta}k}}{|\partial \rho|_\rho^2} + \frac{\rho^k \rho^{\bar{\beta}} \rho_{\bar{\beta}} f_{\bar{w}k}}{|\partial \rho|_\rho^2 \rho_{\bar{w}}} + \frac{\rho^k \rho^{\bar{\beta}} f_{\bar{w}}}{|\partial \rho|_\rho^2} \left[\frac{\rho_{\bar{w}} \rho_{\bar{\beta}k} - \rho_{\bar{\beta}} \rho_{\bar{w}k}}{\rho_{\bar{w}}^2} \right] \\ &\quad - \frac{n \rho^{\bar{k}} f_{\bar{k}}}{|\partial \rho|_\rho^2} + \frac{n f_{\bar{w}}}{\rho_{\bar{w}}}. \end{aligned} \quad (2.18)$$

Here we use summation convention: k runs from 1 to $n+1$ and β runs 1 to n . Simplifying the right hand side, we easily obtain

$$-\square_b f = \left(\rho^{\bar{j}k} - |\partial \rho|_\rho^{-2} \rho^k \rho^{\bar{j}} \right) f_{\bar{j}k} - n |\partial \rho|_\rho^{-2} \rho^{\bar{k}} f_{\bar{k}}, \quad (2.19)$$

which is clearly equivalent to (2.12).

To prove (ii), we notice that

$$X_{\bar{j}\bar{k}}f = \rho_{\bar{k}}f_{\bar{j}} - \rho_{\bar{j}}f_{\bar{k}}. \quad (2.20)$$

Therefore,

$$\begin{aligned} X_{pq}X_{\bar{j}\bar{k}}f &= \rho_q\rho_{\bar{k}}f_{\bar{j}p} + \rho_q\rho_{\bar{k}p}f_{\bar{j}} - \rho_q\rho_{\bar{j}p}f_{\bar{k}} - \rho_q\rho_{\bar{j}}f_{\bar{k}p} \\ &\quad - \rho_p\rho_{\bar{k}}f_{\bar{j}q} - \rho_p\rho_{\bar{k}q}f_{\bar{j}} + \rho_p\rho_{\bar{j}}f_{\bar{k}q} + \rho_p\rho_{\bar{j}q}f_{\bar{k}}. \end{aligned} \quad (2.21)$$

Contracting both side with $\rho^{p\bar{k}}\rho^{q\bar{j}}$, using (i), we easily obtain (ii). The proof is complete. \square

3. AN ESTIMATE FOR EIGENVALUES AND PROOF OF THEOREM 1.1

We denote by $S: L^2(M) \rightarrow \ker \square_b (= \ker \bar{\partial}_b)$ the Szegő orthogonal projection with respect to the volume measure $dv := \theta \wedge (d\theta)^n$. It is well-known that if M is embeddable, then $\text{Spec}(\square_b)$ consists of zero and a sequence of point eigenvalues $\{\lambda_k\}$ increasing to infinity. The positive eigenvalues of \square_b are of finite multiplicity and eigenfunctions are smooth [3, 6]. Furthermore, we have the following orthogonal decomposition:

$$L^2(M, dv) = \bigoplus_{k=0}^{\infty} E_k, \quad E_0 = \ker \square_b. \quad (3.1)$$

Note that E_0 is of infinite dimension.

Theorem 3.1. *Let (M, θ) be an embedded compact strictly pseudoconvex pseudohermitian manifold and $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$ the eigenvalues for \square_b . Define*

$$m(a) = \inf \left\{ \left| a - \frac{1}{\lambda_k} \right|^2 : k \in \mathbb{N} \right\}, \quad M(a) = \sup \left\{ \left| a - \frac{1}{\lambda_k} \right|^2 : k \in \mathbb{N} \right\}. \quad (3.2)$$

Then for any $a \in \mathbb{R}$, any function $u \notin \ker \square_b$,

$$(m(a) - a^2)\|\square_b u\|^2 \leq \|u - S(u)\|^2 - \int_M |\bar{\partial}_b u|^2 \leq (M(a) - a^2)\|\square_b u\|^2. \quad (3.3)$$

Proof. Let E_k be the eigenspace of \square_b associated to the eigenvalue λ_k . Then $m_k = \dim(E_k) < \infty$. Let $\{f_{k,j}\}_{j=1}^{m_k}$ be an orthonormal basis for E_k . For any k, ℓ , using integration by parts, we obtain

$$\int_M (\square_b u - \lambda_k u) \bar{f}_{k,\ell} = \int_M (u \overline{\square_b f_{k,\ell}} - \lambda_k u \bar{f}_{k,\ell}) = \int_M u (\overline{\square_b f_{k,\ell}} - \lambda_k \bar{f}_{k,\ell}) = 0. \quad (3.4)$$

This implies that for any real number a ,

$$\langle u - a\square_b u, f_{k\ell} \rangle = -(a - 1/\lambda_k) \langle \square_b u, f_{k\ell} \rangle. \quad (3.5)$$

Therefore, since $\square_b u \in (\ker \square_b)^\perp$,

$$M(a) \|\square_b u\|^2 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} M(a) |\langle \square_b u, f_{k,\ell} \rangle|^2 \quad (3.6)$$

$$\geq \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} \left| a - \frac{1}{\lambda_k} \right|^2 |\langle \square_b u, f_{k,\ell} \rangle|^2 \quad (3.7)$$

$$= \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} |\langle u - a \square_b u, f_{k,\ell} \rangle|^2 \quad (3.8)$$

$$= \|u - a \square_b u\|^2 - \|S(u - a \square_b u)\|^2 \quad (3.9)$$

$$= \|u\|^2 + a^2 \|\square_b u\|^2 - 2a \int_M \bar{u} \square_b u - \|S(u)\|^2. \quad (3.10)$$

Here we have used $\|S(u - a \square_b u)\|^2 = \|S(u)\|^2$. We conclude that

$$(M(a) - a^2) \|\square_b u\|^2 \geq \|u - S(u)\|^2 - 2a \int_M |\bar{\partial}_b u|^2. \quad (3.11)$$

This proves the second inequality. The first inequality can be proved similarly. \square

The following two corollaries are undoubtedly known, but we can not find in the literature.

Corollary 3.2. *Let (M, θ) be as in Theorem 3.1, then for any function u ,*

$$\lambda_1 = \inf \left\{ \|\square_b u\|^2 : \int_M |\bar{\partial}_b u|^2 = 1 \right\} = \inf \left\{ \int_M |\bar{\partial}_b u|^2 : \|u - S(u)\|^2 = 1 \right\}. \quad (3.12)$$

Proof. For any $a > \frac{1}{\lambda_1}$, we have

$$m(a) = \left| a - \frac{1}{\lambda_1} \right|^2. \quad (3.13)$$

From Theorem 3.1, we have for any u with $\int_M |\bar{\partial}_b u|^2 = 1$,

$$\left[\left| a - \frac{1}{\lambda_1} \right|^2 - a^2 \right] \|\square_b u\|^2 \leq \|u - S(u)\|^2 - 2a. \quad (3.14)$$

This is equivalent to

$$\left(\frac{1}{\lambda_1} \right)^2 - 2a \left(\frac{1}{\lambda_1} - \frac{1}{\|\square_b u\|^2} \right) \leq \frac{\|u - S(u)\|^2}{\|\square_b u\|^2}. \quad (3.15)$$

Letting $a \rightarrow +\infty$, we easily obtain

$$\lambda_1 \leq \|\square_b u\|^2. \quad (3.16)$$

Since u is arbitrary, we conclude that

$$\lambda_1 \leq \inf \left\{ \|\square_b u\|^2 : \int_M |\bar{\partial}_b u|^2 = 1 \right\}. \quad (3.17)$$

The reverse inequality is trivial. This completes the proof of the first equality.

To prove the second we take $a = \frac{1}{2} \frac{1}{\lambda_1}$ and notice that $M(a) = a^2$. Then from Theorem 3.1, we deduce that for any u satisfying $\|u - S(u)\|^2 = 1$,

$$0 = (M(a) - a^2) \|\square_b u\|^2 \geq \|u - S(u)\|^2 - 2a \int_M |\bar{\partial}_b u|^2 = 1 - 2a \int_M |\bar{\partial}_b u|^2. \quad (3.18)$$

Hence,

$$\lambda_1 = \frac{1}{2a} \leq \int_M |\bar{\partial}_b u|^2. \quad (3.19)$$

The proof of the reverse inequality is simple and omitted. \square

Corollary 3.3. *Let (M, θ) be as in Theorem 3.1, then for any function u ,*

$$\|u - S(u)\| \cdot \|\square_b u\| \geq \int_M |\bar{\partial}_b u|^2. \quad (3.20)$$

Proof. Without loss of generality, we may assume that $\int_M |\bar{\partial}_b u|^2 = 1$. For each k , we take $a_k = \frac{1}{2} \left(\frac{1}{\lambda_k} + \frac{1}{\lambda_{k+1}} \right)$. Clearly,

$$m(a_k) = \left| a_k - \frac{1}{\lambda_k} \right|^2 = \left| a_k - \frac{1}{\lambda_{k+1}} \right|^2 \quad (3.21)$$

By Theorem 3.1, we have

$$\left[\left| a_k - \frac{1}{\lambda_k} \right|^2 - a_k^2 \right] \|\square_b u\|^2 \leq \|u - S(u)\|^2 - 2a_k. \quad (3.22)$$

By direct calculation, we have that

$$\left(\frac{1}{\lambda_k} - \frac{1}{\|\square_b u\|^2} \right) \left(\frac{1}{\lambda_{k+1}} - \frac{1}{\|\square_b u\|^2} \right) \geq \frac{1}{\|\square_b u\|^4} - \frac{\|u - S(u)\|^2}{\|\square_b u\|^2}. \quad (3.23)$$

By Corollary 3.2, $\lambda_1 \leq \|\square_b u\|^2$. Moreover, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. We deduce that there exists k_0 such that

$$\lambda_{k_0} \leq \|\square_b u\|^2 < \lambda_{k_0+1}. \quad (3.24)$$

Therefore, (3.23) with $k = k_0$ implies that

$$\frac{1}{\|\square_b u\|^4} - \frac{\|u - S(u)\|^2}{\|\square_b u\|^2} \leq 0. \quad (3.25)$$

This completes the proof. \square

Proposition 3.4. *Let (M, θ) be compact, strictly pseudoconvex pseudohermitian manifold. If there is a smooth non-CR function f on M such that $|\square_b f|^2 \leq B(z) \operatorname{Re} \bar{f} \square_b f$ for some non-negative function B on M , then*

$$\lambda_1 \leq \max_M B(z). \quad (3.26)$$

If the equality holds, then B must be a constant.

Proof. Since $|\square_b f|^2 \leq B(z) \operatorname{Re} \bar{f} \square_b f$, by Corollary 3.2,

$$\lambda_1 \int_M \bar{f} \square_b f \leq \int_M |\square_b f|^2 \leq \int_M B(z) \operatorname{Re} (\bar{f} \square_b f). \quad (3.27)$$

By the Mean Value Theorem of the integral, there is $z_0 \in M$ such that

$$0 \leq \int_M (B - \lambda_1) \operatorname{Re} (\bar{f} \square_b f) = (B(z_0) - \lambda_1) \int_M \bar{f} \square_b f = (B(z_0) - \lambda_1) \int_M |\bar{\partial}_b f|^2. \quad (3.28)$$

This implies

$$\lambda_1 \leq B(z_0) \leq \max_M B(z). \quad (3.29)$$

It is clear that if $\lambda_1 = \max_M B$ then B is a constant. \square

We end this section by proving the Theorem 1.1.

Proof of Theorem 1.1. By the condition (1.4) and the expression for Kohn-Laplacian given by (2.12), we have

$$\square_b \rho_j = \tilde{\Delta} \rho_j + n |\partial \rho|_\rho^{-2} \rho^{\bar{k}} \rho_{j\bar{k}} = \tilde{\Delta} \rho_j + n |\partial \rho|_\rho^{-2} \rho_j. \quad (3.30)$$

Then

$$|\square_b \rho_j|^2 = \frac{n}{|\partial \rho|_\rho^2} \operatorname{Re} \left(\rho_{\bar{j}} \square_b \rho_j + \rho_{\bar{j}} \tilde{\Delta} \rho_j + \frac{1}{n} |\partial \rho|_\rho^2 |\tilde{\Delta} \rho_j|^2 \right) \leq \frac{n}{|\partial \rho|_\rho^2} \operatorname{Re} (\rho_{\bar{j}} \square_b \rho_j) \quad (3.31)$$

Applying Proposition 3.4 with $B(z) = n |\partial \rho|_\rho^{-2}$, we obtain

$$\lambda_1 \leq n \max_M |\partial \rho|_\rho^{-2}. \quad (3.32)$$

The equality holds only if $|\partial \rho|_\rho$ is a constant on M . The proof of Theorem 1.1 is complete. \square

4. PROOF OF THEOREM 1.2

The following theorem gives a sharp upper bound for $\lambda_1(\square_b)$ in terms of the eigenvalues of the complex Hessian matrix $[\rho_{j\bar{k}}]$ and the length of $\partial\rho$. This theorem implies the estimate in Theorem 1.2.

Theorem 4.1. *Let ρ be a smooth strictly plurisubharmonic function defined on an open set U of \mathbb{C}^{n+1} , M a compact connected regular level set of ρ , and λ_1 the first positive eigenvalue of \square_b on M . Let $r(z)$ be the spectral radius of the matrix $[\rho^{j\bar{k}}(z)]$ and $s(z) = \text{trace}[\rho^{j\bar{k}}] - r(z)$. Then*

$$\lambda_1 \leq \frac{n^2 \int_M r(z) |\partial\rho|_\rho^{-2}}{\int_M s(z)}. \quad (4.1)$$

Here the spectral radius of a square matrix is the maximum of the moduli of its eigenvalues.

Proof. First, we define

$$C_j = \int_M \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^4}, \quad D_j = \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^2} \right). \quad (4.2)$$

From Proposition 2.1, we can compute

$$\square_b \bar{z}^j = n |\partial\rho|^{-2} \rho^{\bar{j}}. \quad (4.3)$$

Therefore,

$$\|\square_b \bar{z}^j\|^2 = n^2 \int_M \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^4} = n^2 C_j. \quad (4.4)$$

We can also compute

$$|\bar{\partial}_b \bar{z}^j|^2 = \delta_{j\alpha} \delta_{j\beta} \left(\rho^{\alpha\bar{\beta}} - \frac{\rho^\alpha \rho^{\bar{\beta}}}{|\partial\rho|^2} \right) = \rho^{j\bar{j}} - \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|^2}. \quad (4.5)$$

Here without loss of generality, we assume $j \neq n+1$. Therefore,

$$\int_M |\bar{\partial}_b \bar{z}^j|^2 = D_j. \quad (4.6)$$

Thus, from Corollary 3.2 above, we obtain for all j

$$\lambda_1 \leq n^2 C_j / D_j. \quad (4.7)$$

Next, observe that $1/r(z)$ is the smallest eigenvalue of the Hermitian matrix $[\rho_{j\bar{k}}(z)]$, and thus, for all $(n+1)$ -vector v^j ,

$$\frac{1}{r(z)} \sum_{j=1}^{n+1} |v^j|^2 \leq v^j \rho_{j\bar{k}} v^{\bar{k}}. \quad (4.8)$$

Plugging $v^j = \rho^j$ into the inequality, we easily obtain $\sum_{j=1}^{n+1} |\rho^j|^2 \leq r(z) |\partial \rho|_\rho^2$. Consequently

$$\sum_j C_j = \sum_{j=1}^{n+1} \int_M \frac{|\rho^j|^2}{|\partial \rho|_\rho^4} \leq \int_M r(z) |\partial \rho|^{-2}, \quad (4.9)$$

and therefore,

$$\sum_j D_j = \sum_{j=1}^{n+1} \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial \rho|_\rho^2} \right) \geq \int_M [\text{trace}[\rho^{j\bar{k}}] - r(z)] = \int_M s(z). \quad (4.10)$$

Thus, from (4.7), (4.9), and (4.10), we obtain

$$\lambda_1 \leq n^2 \min_j (C_j / D_j) \leq \frac{n^2 \sum_j C_j}{\sum_j D_j} = \frac{n^2 \int_M r(z) |\partial \rho|_\rho^{-2}}{\int_M s(z)}. \quad (4.11)$$

The proof is complete. \square

Proof of Theorem 1.2. Since $\rho_{j\bar{k}} = \delta_{jk}$, we have $r(z) = 1$ and $s(z) = n$. Therefore, by Theorem 4.1,

$$\lambda_1 \leq \frac{n^2 \int_M r(z) |\partial \rho|_\rho^{-2}}{\int_M s(z)} = \frac{n}{v(M)} \int_M |\partial \rho|_\rho^{-2}. \quad (4.12)$$

which proves the inequality.

Next we suppose that $\lambda_1 = \frac{n}{v(M)} \int_M |\partial \rho|_\rho^{-2}$. We shall show that $|\partial \rho|_\rho^2$ is constant along M . Put

$$b_j = n^{-1} \square_b \bar{z}^j = |\partial \rho|^{-2} \rho_j. \quad (4.13)$$

Then by inspecting the proof of Theorem 3.1 above, in particular, the estimate (3.6), we have for all j ,

$$\langle b_j, f_{k,\ell} \rangle = 0, \quad \text{for all } \ell, \text{ for all } k \neq 1. \quad (4.14)$$

Thus, $b_j \perp \ker \square_b$ and (4.14) imply that $b_j \in E_1$, the eigenspace corresponding to λ_1 . Therefore,

$$\square_b b_j = \lambda_1 b_j. \quad (4.15)$$

Recall that $\square_b \bar{z}^j = n b_j$. We then deduce that

$$\square_b \left[\bar{z}^j - \frac{n}{\lambda_1} \frac{\rho_j}{|\partial \rho|^2} \right] = 0. \quad (4.16)$$

Hence, $\bar{z}^j - n \rho^{\bar{j}} / (\lambda_1 |\partial \rho|^2)$ restricted to M is a CR function. Since $X_{\bar{ik}}$ is a tangential CR vector fields on M , we have

$$X_{\bar{ik}} \left[\bar{z}^j - \frac{n}{\lambda_1} \frac{\rho_j}{|\partial \rho|^2} \right] = 0. \quad (4.17)$$

By direct calculation, this is equivalent to

$$\frac{n}{\lambda_1} \rho_j X_{\bar{l}k} (|\partial\rho|^2) / |\partial\rho|^4 = \left(1 - \frac{n}{\lambda_1 |\partial\rho|^2}\right) (\rho_{\bar{l}} \delta_{jk} - \rho_{\bar{k}} \delta_{jl}). \quad (4.18)$$

Since M is compact, there exists point $x \in M$ such that

$$|\partial\rho(x)|^2 = \max_M |\partial\rho|^2. \quad (4.19)$$

At the maximum point x , we also have $X_{\bar{l}k} |\partial\rho|^4 = 0$. Thus,

$$\left[1 - \frac{n}{\lambda_1 |\partial\rho|^2}\right] (\rho_{\bar{l}} \delta_{jk} - \rho_{\bar{k}} \delta_{jl}) = 0 \quad \text{at } x. \quad (4.20)$$

Since $\partial\rho(x) \neq 0$, we can assume that $\rho_{\bar{1}}(x) \neq 0$. Taking $j = k = 2$, we have at x

$$1 - \frac{n}{\lambda_1 |\partial\rho|^2} = 0. \quad (4.21)$$

Therefore,

$$\min |\partial\rho|^{-2} = |\partial\rho(x)|^{-2} = \frac{\lambda_1}{n} = \frac{1}{v(M)} \int_M |\partial\rho|^{-2}. \quad (4.22)$$

As the right most term is the average of $|\partial\rho|^{-2}$ on M , we deduce from above that $|\partial\rho|^{-2}$ must be constant on M .

Finally, suppose that $|\partial\rho|^2$ is constant along M and ρ extends to the domain bounded by M and satisfies $\rho_{j\bar{k}} = \delta_{jk}$ on the domain. We shall show in the lemma below that M must be a sphere and complete the proof of Theorem 1.2. \square

Lemma 4.2. *Suppose M be a compact, connected regular level set of ρ which bounds a domain D and satisfies $\rho_{j\bar{k}} = \delta_{jk}$ on D . If $|\partial\rho|^2$ is constant on M , then M must be a sphere.*

Proof of Lemma 4.2. The proof is an application of Serrin's theorem [23, Theorem 1]. Let D be the domain with M as its boundary. Define $u = \rho - \nu$ on a neighborhood of \bar{D} . Since M is smooth and the function u satisfies $\Delta u = -4(n+1)$ in D , $u = 0$ on ∂D , and the normal derivative $\partial u / \partial \mathbf{n} = 2|\partial\rho|$ is a constant on ∂D by assumption we can apply the Serrin's theorem to conclude that M is a standard sphere. \square

We end this section by the following example which gives a sharp upper bound on the family of compact level sets of Kähler potentials of Fubini-Study metric. This example also shows that the condition (1.4) in Theorem 1.1 can not be relaxed.

Example 4.3. Let ρ be a strictly plurisubharmonic function of the form

$$\rho(Z) = \log(1 + \|Z\|^2) + \psi(Z, \bar{Z}), \quad (4.23)$$

where ψ is a real-valued pluriharmonic function. We suppose that ρ is defined and proper in some domain $U \subset \mathbb{C}^{n+1}$ (e.g., if $\psi = -\log|z_1|$, then ρ is defined and proper on $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n$).

Observe that

$$\rho_{j\bar{k}} = \frac{1}{1 + \|Z\|^2} \left(\delta_{jk} - \frac{\bar{z}^j z^k}{1 + \|Z\|^2} \right), \quad \rho^{j\bar{k}} = (1 + \|Z\|^2) (\delta_{jk} + \bar{z}^k z^j), \quad (4.24)$$

By a routine calculation, we see that the characteristic polynomial of $[\rho^{j\bar{k}}]$ is

$$P_{[\rho^{j\bar{k}}]}(\lambda) = (1 + \|Z\|^2 - \lambda)^n [(1 + \|Z\|^2)^2 - \lambda]. \quad (4.25)$$

Thus, the spectral radius of $[\rho^{j\bar{k}}]$ is $r(Z) = (1 + \|Z\|^2)^2$ and $s(Z) = \text{trace}[\rho^{j\bar{k}}] - r(Z) = n(1 + \|Z\|^2)$. By Theorem 4.1, if M is a compact, connected, regular level set of ρ , then

$$\lambda_1 \leq \frac{n \int_M (1 + \|Z\|^2)^2 |\partial\rho|_\rho^{-2}}{\int_M (1 + \|Z\|^2)} \leq n \max_M (1 + \|Z\|^2) |\partial\rho|_\rho^{-2}. \quad (4.26)$$

Notice that if $\psi = 0$ and then $M_\nu := \rho^{-1}(\nu)$ is the sphere $\|Z\|^2 = e^\nu - 1$ with

$$\theta = ie^{-\nu} \sum_{j=1}^{n+1} (z^j d\bar{z}^j - \bar{z}^j dz^j). \quad (4.27)$$

Moreover, $|\partial\rho|_\rho^2 = e^\nu - 1$ on M_ν and $\lambda_1 = ne^\nu/(e^\nu - 1)$. Therefore, the condition (1.4) in Theorem 1.1 can not be relaxed.

5. THE REAL ELLIPSOIDS: PROOF OF COROLLARY 1.3

The proof of Corollary 1.3 follows from Theorem 4.1 and the proposition below.

Proposition 5.1. *Let $Q(Z)$ be a quadratic polynomial and let $M_\nu = \rho^{-1}(\nu)$ be a compact regular level set of ρ , where ρ is given by*

$$\rho(Z) = \sum_{k=1}^{n+1} |z^k|^2 + 2\text{Re } Q(Z) \quad (5.1)$$

Then

$$C_\nu := \frac{1}{v(M_\nu)} \int_{M_\nu} |\partial\rho|^{-2} = \frac{1}{\nu}. \quad (5.2)$$

Proof. We observe that

$$\text{Re} \sum_{j=1}^{n+1} z^j \rho^{\bar{j}} = \sum_{j=1}^{n+1} |z^j|^2 + \text{Re} \sum_{j=1}^{n+1} z^j Q_j = \nu - 2\text{Re } Q + \text{Re} \sum_{j=1}^{n+1} z^j Q_j. \quad (5.3)$$

As Q is a quadratic polynomial, we can check that $\sum_{j=1}^{n+1} z^j Q_j = 2Q$. Hence

$$\operatorname{Re} \sum_{j=1}^{n+1} z^j \bar{\rho}^j = \nu \quad \text{on } M_\nu. \quad (5.4)$$

Therefore,

$$\int_{M_\nu} |\partial \rho|^{-2} = \operatorname{Re} \frac{1}{\nu} \sum_{j=1}^{n+1} \int_{M_\nu} \frac{z^j \rho_j}{|\partial \rho|^2} = \operatorname{Re} \frac{1}{\nu} \sum_{j=1}^{n+1} \int_{M_\nu} \frac{(z^j + \overline{Q_j}) \rho_j}{|\partial \rho|^2} = \frac{v(M)}{\nu}, \quad (5.5)$$

Here we use

$$\int_{M_\nu} \frac{\overline{Q_j} \rho_j}{|\partial \rho|^2} = \frac{1}{n} \int_{M_\nu} \overline{Q_j} \square_b \bar{z}^j = \frac{1}{n} \int_{M_\nu} \bar{z}^j \square_b \overline{Q_j} = 0. \quad (5.6)$$

Hence, $C_\nu = \frac{1}{\nu}$. \square

Proof of Corollary 1.3. From Theorem 1.2 and Proposition 5.1, we have

$$\lambda_1 \leq nC_\nu = \lambda_1(\sqrt{\nu} \mathbb{S}^{2n+1}). \quad (5.7)$$

Also from Theorem 1.2, we see that the equality occurs if and only if M must be the sphere and hence the proof is complete. However, we provide here an elementary proof of this last step below. Notice that

$$Q(Z) = \sum_{k,j=1}^n q_{jk} z_k z_j \quad (5.8)$$

and $Q = [q_{jk}]$ is $n \times n$ symmetric matrix. By an well-known factorization theorem (see [14, Section 3.5]), we can write $Q = U^t \Lambda U$, where U is a unitary matrix and $\Lambda = \operatorname{Diag}(A_1, \dots, A_{n+1})$ is a diagonal matrix with $A_j \geq 0$. We make a holomorphic unitary change of variables $W = UZ$, then

$$\rho(Z) = \|W\|^2 + \operatorname{Re} \sum_{j=1}^{n+1} A_j w_j^2 \quad (5.9)$$

Without loss of generality, one may assume that $\rho(Z) = \|Z\|^2 + \operatorname{Re} \sum_{j=1}^{n+1} |z_j|^2$. Since M_ν is bounded, it is easy to see that $A_j < 1$ for $1 \leq j \leq n+1$. Notice that on M_ν

$$c = |\partial \rho|^2 = \nu + \sum_{j=1}^{n+1} A_j^2 |z_j|^2 + \operatorname{Re} \sum_{j=1}^{n+1} A_j z_j^2 = 2\nu - |Z|^2 + \sum_{j=1}^{n+1} A_j |z_j|^2 \quad (5.10)$$

If the equality occurs, then $|\partial \rho|^2$ is a constant along M_ν . Restricting $z = \lambda \mathbf{e}_j \in M_\nu$, one has $|z_j|^2$ must be a constant. This can not be true unless $A_j = 0$. This proves $\rho(Z) = \|Z\|^2$, and M_ν is a sphere centered at 0 with radius $\sqrt{\nu}$. \square

6. CR MANIFOLDS IMBEDDED INTO THE SPHERE

Theorem 6.1. *Let ρ be a smooth strictly plurisubharmonic function defined on an open set U of \mathbb{C}^{n+1} , M a compact connected regular level set of ρ , and λ_1 the first positive eigenvalue of \square_b on M . Suppose that ρ can be written as*

$$\rho = \sum_{\mu=1}^N |f^{(\mu)}(z)|^2 + \psi(z, \bar{z}). \quad (6.1)$$

where $f^{(\mu)}(z)$ holomorphic functions, $\psi(z, \bar{z})$ pluriharmonic on M on a neighborhood of M . Then

$$\lambda_1 \leq \frac{n}{v(M)} \int_M |\partial\rho|_\rho^{-2} \theta \wedge (d\theta)^n. \quad (6.2)$$

Remark 6.2. (a) Theorem 6.1 applies to all CR manifolds imbedded into a sphere.

Indeed, if M is a strictly pseudoconvex CR manifold and $H = (f^{(1)}, f^{(2)}, \dots, f^{(N)})$ is an holomorphic immersion sending M into the sphere in \mathbb{C}^N , then M is a level set $\rho^{-1}(1)$ with ρ is given by (6.1) and $\psi = 0$.

(b) We can interpret Theorem 6.1 as a necessary condition, involving only the restrictions of $i\partial\rho$ and $|\partial\rho|$ to a compact regular level set of ρ , for a positive strictly plurisubharmonic function ρ to be a sum of “hermitian squares”.

Proof. Observe that since $f^{(\mu)}$ are holomorphic and ψ is pluriharmonic,

$$\rho_{l\bar{k}} = \sum_{\mu=1}^N \bar{f}_k^{(\mu)} f_l^{(\mu)}. \quad (6.3)$$

On the other hand, by Proposition 2.1, $\square_b \bar{f}^{(\mu)} = n|\partial\rho|_\rho^{-2} \rho^{\bar{k}} \bar{f}_k^{(\mu)}$. Therefore,

$$|\square_b \bar{f}^{(\mu)}|^2 = n^2 |\partial\rho|_\rho^{-4} \rho^{\bar{k}} \bar{f}_k^{(\mu)} \rho^l f_l^{(\mu)}. \quad (6.4)$$

Summing over $\mu = 1, 2, \dots, N$, we obtain

$$\sum_{\mu=1}^N |\square_b \bar{f}^{(\mu)}|^2 = n^2 |\partial\rho|_\rho^{-4} \rho^{\bar{k}} \rho^l \sum_{\mu=1}^N \bar{f}_k^{(\mu)} f_l^{(\mu)} = n^2 |\partial\rho|_\rho^{-4} \rho^{\bar{k}} \rho^l \rho_{l\bar{k}} = n^2 |\partial\rho|_\rho^{-2}. \quad (6.5)$$

Next, observe that by (6.3) and (2.6) (assuming $\rho_w \neq 0$)

$$\begin{aligned}
\sum_{\mu=1}^N Z_{\bar{\gamma}} \bar{f}^{(\mu)} Z_{\sigma} f^{(\mu)} &= \sum_{\mu=1}^N \left(\bar{f}_{\bar{\gamma}}^{(\mu)} - \frac{\rho_{\bar{\gamma}}}{\rho_{\bar{w}}} \bar{f}_{\bar{w}}^{(\mu)} \right) \left(f_{\sigma}^{(\mu)} - \frac{\rho_{\sigma}}{\rho_w} f_w^{(\mu)} \right) \\
&= \sum_{\mu=1}^N \bar{f}_{\bar{\gamma}}^{(\mu)} f_{\sigma}^{(\mu)} - \frac{\rho_{\bar{\gamma}}}{\rho_{\bar{w}}} \sum_{\mu=1}^N \bar{f}_{\bar{w}}^{(\mu)} f_{\sigma}^{(\mu)} - \frac{\rho_{\sigma}}{\rho_w} \sum_{\mu=1}^N f_w^{(\mu)} \bar{f}_{\bar{\gamma}}^{(\mu)} + \frac{\rho_{\bar{\gamma}} \rho_{\sigma}}{|\rho_w|^2} \sum_{\mu=1}^N f_w^{(\mu)} \bar{f}_{\bar{w}}^{(\mu)} \\
&= \rho_{\sigma \bar{\gamma}} - \frac{\rho_{\bar{\gamma}} \rho_{\sigma \bar{w}}}{\rho_{\bar{w}}} - \frac{\rho_{\sigma} \rho_{\bar{\gamma} w}}{\rho_w} + \frac{\rho_{\bar{\gamma}} \rho_{\sigma} \rho_{w \bar{w}}}{|\rho_w|^2} \\
&= h_{\sigma \bar{\gamma}}.
\end{aligned} \tag{6.6}$$

Therefore,

$$\sum_{\mu=1}^N |\bar{\partial}_b \bar{f}^{(\mu)}|^2 = h^{\sigma \bar{\gamma}} \sum_{\mu=1}^N Z_{\bar{\gamma}} \bar{f}^{(\mu)} Z_{\sigma} f^{(\mu)} = n. \tag{6.7}$$

Applying Corrolary 3.2, we obtain,

$$\lambda_1 \leq \min_{\mu} \frac{\|\square_b \bar{f}^{(\mu)}\|^2}{\|\bar{\partial}_b \bar{f}^{(\mu)}\|^2} \leq \frac{\sum_{\mu=1}^N \|\square_b \bar{f}^{(\mu)}\|^2}{\sum_{\mu=1}^N \|\bar{\partial}_b \bar{f}^{(\mu)}\|^2} = \frac{n}{v(M)} \int_M |\partial \rho|^{-2}. \tag{6.8}$$

The proof is complete. \square

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